Math 4650

Topic 1a –

Derivation of real number properties from field and order properties

In this <u>optional</u> topic I will show you how we can use the field properties (A1)-(A5), (M1)-(M5), (D1) and the order properties (O1)-(O4) to prove various algebraic and order properties of $\mathbb{R}$.

We will follow the disussion from Rudin's "Principles of Mathematical Analysis"

<u>Theorem:</u> Let $x, y, z \in \mathbb{R}$.

1. If $x+y = x+z$, then $y = z$.
2. If $x+y = x$, then $y = 0$.
3. If $x+y = 0$, then $y = -x$.
4. $-(-x) = x$
5. If $x \neq 0$ and $xy = xz$, then $y = z$
6. If $x \neq 0$ and $xy = x$, then $y = 1$
7. If $x \neq 0$ and $xy = 1$, then $y = x^{-1}$
8. If $x \neq 0$ then $(x^{-1})^{-1} = x$
9. $0x = 0$
10. If $x \neq 0$ and $y \neq 0$, then $xy \neq 0$

inverses are unique

(11) $(-x)y = -(xy) = x(-y)$

(12) $(-x)(-y) = xy$

proof:

(1) Suppose $x+y = x+z$.

Then,

$$y \underset{(A4)}{=} 0+y \underset{(A5)}{=} (-x+x)+y \underset{(A3)}{=} -x+(x+y)$$

$$= -x+(x+z) \underset{(A3)}{=} (-x+x)+z$$

$$\underset{(A5)}{=} 0+z \underset{(A4)}{=} z$$

(2) Plug $z=0$ into part 1.

(3) Plug $z=-x$ into part 1.

(4) Since $(-x)+x = 0$

We can apply part 3 to get $x = -(-x)$.

(5) – (8): You try, they are similar to (1) – (4).

(9) We have

$$0x \underset{(A4)}{=} (0+0)x \underset{(D1)}{=} 0x + 0x$$

Then,

$$-0x + 0x = -0x + (0x + 0x)$$

So by (A5) and (A3) we get

$$0 = (-0x + 0x) + 0x$$

By (A5) we get

$$0 = 0 + 0x$$

By (A4) we get

$$0 = 0x.$$

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⑩ Let $x, y \in \mathbb{R}$ with $x \neq 0$ and $y \neq 0$. Let's show this implies that $xy \neq 0$.

Suppose $xy = 0$.

Then

$$1 \underset{(M4)}{=} 1 \cdot 1 \underset{(M5)}{=} (x^{-1}x)(y^{-1}y) \underset{(M3)}{=} x^{-1}(xy^{-1})y \underset{(M2)}{=} x^{-1}(y^{-1}x)y$$

$$\underset{(M3)}{=} (x^{-1}y^{-1})(xy) = x^{-1}y^{-1}0 \underset{(\text{part } 9)}{=} 0$$

We get the contradiction $1 = 0$.

Hence we must have $xy \neq 0$.

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(11) We have that

$$(-x)y + xy \underset{(D1)}{=} (-x+x)\,y \underset{(A5)}{=} 0y \underset{(\text{part } 9)}{=} 0$$

Thus, by part 3 we get

$$(-x)y = -(xy).$$

Similarly,

$$x(-y) + xy = x(-y+y) = x\,0 = 0$$

So, $x(-y) = -(xy)$.

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(12) We have that

$$(-x)(-y) \underset{(\text{part } 11)}{=} -(x(-y)) \underset{(\text{part } 11)}{=} -(-(xy)) \underset{(\text{part } 4)}{=} xy$$

$\blacksquare$

Theorem: Let $x, y \in \mathbb{R}$.

Then:

1. If $x > 0$, then $-x < 0$.
2. If $x < 0$, then $-x > 0$.
3. If $x > 0$ and $y < z$, then $xy < xz$.
4. If $x < 0$ and $y < z$, then $xy > xz$.
5. If $x \neq 0$, then $x^2 > 0$.
6. $1 > 0$
7. If $0 < x < y$, then $0 < y^{-1} < x^{-1}$.

Proof:

① If $x > 0$, then

$$0 \underset{(AS)}{=} -x + x \underset{(O3)}{>} -x + 0 = -x$$

Thus, $0 > -x$.

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② If $x < 0$, then

$$0 \underset{(AS)}{=} -x + x \underset{(O3)}{<} -x + 0 = -x.$$

Thus, $0 < -x$.

(3) Suppose $x > 0$ and $y < z$.

Then,

$$z - y \underset{(O3)}{>} y - y = 0$$

So,

$$z - y > 0.$$

Thus, by (O4) we get

$$x(z-y) > 0$$

So,

$$\begin{aligned}
xz &\overset{(A4)}{=} xz + 0 \\
&\underset{(A5)}{=} xz + (-xy + xy) \\
&\underset{(A3)}{=} (xz - xy) + xy \\
&\underset{(D1)}{=} x(z-y) + xy \\
&\underset{(O3)}{>} 0 + xy \\
&\underset{(A4)}{=} xy
\end{aligned}$$

Thus, $xz > xy$.

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(4) Suppose $x < 0$ and $y < z$.

Since $x<0$ we have $-x > 0$. ← (part 2)

Since $y<z$ we have $-y+y < -y+z$ ← (O3)

So, $0 < z-y$ ← (A4) and (A2)

Then,

$$-[x(z-y)] \underset{\substack{\uparrow \\ \text{(previous theorem)}}}{=} (-x)(z-y) \underset{\substack{\uparrow \\ (O4)}}{>} 0$$

Thus, $x(z-y) < 0$ by part 1.

So, $xz - xy < 0$. ← (D1 and previous theorem)

Thus, $xz - xy + xy < 0 + xy$ ← (O3)

So, $xz + 0 < xy$ ← (A4, A5)

Thus, $xz < xy$. ← (A4)

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(5) Suppose $x \neq 0$.

If $x > 0$, then $x \cdot x > 0 \cdot 0$ by (O4).

So if $x > 0$, then $x^2 > 0$. ← $\boxed{\text{$0 \cdot 0 = 0$ by prev. theorem}}$

If $x<0$, then $-x>0$ by part 2.
Hence if $-x<0$, then $(-x)^2>0$.
But $(-x)^2=(-x)(-x)=x^2$ by prev. thm.
Thus, if $x<0$, then $x^2>0$.

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6 Plug $x=1$ into part 5
to get that $1^2>0$. (M4)
Thus, $1>0$

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7 Suppose $0<x<y$.
By (O2) we know $0<y$.

We know $y \cdot y^{-1} = 1 > 0$
(M5) (part 6)

Since $y>0$ and $yy^{-1}>0$ we must
have that $y^{-1}>0$ by (O4).

Similarly $x^{-1}>0$.
Since $x<y$ we know $x(x^{-1}y^{-1})<y(x^{-1}y^{-1})$
Thus, $(xx^{-1})y^{-1}<y(y^{-1}x^{-1})$ by (M3).
So, $1y^{-1}<(yy^{-1})x^{-1}$ by (M3, M5)
Thus, $y^{-1}<1x^{-1}$ by (M4, M5)

So, $y^{-1} < x^{-1}$ by (M4).

Thus, $0 < y^{-1} < x^{-1}$.

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