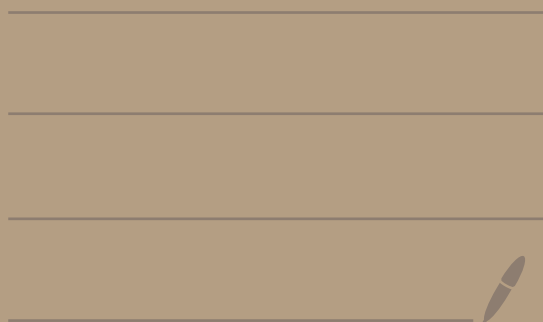


Math 4650

Topic 1a -

Derivation of real number  
properties from field and  
order properties



In this optional topic I will show you how we can use the field properties (A1)-(A5), (M1)-(M5), (D1) and the order properties (O1)-(O4) to prove various algebraic and order properties of  $\mathbb{R}$ .

We will follow the discussion from Rudin's "Principles of Mathematical Analysis"

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Theorem: Let  $x, y, z \in \mathbb{R}$ .

- ① If  $x+y = x+z$ , then  $y=z$ .
- ② If  $x+y = x$ , then  $y=0$ .
- ③ If  $x+y = 0$ , then  $y=-x$ .
- ④  $-(-x) = x$
- ⑤ If  $x \neq 0$  and  $xy = xz$ , then  $y=z$
- ⑥ If  $x \neq 0$  and  $xy = x$ , then  $y=1$
- ⑦ If  $x \neq 0$  and  $xy = 1$ , then  $y = x^{-1}$
- ⑧ If  $x \neq 0$  then  $(x^{-1})^{-1} = x$
- ⑨  $0x = 0$
- ⑩ If  $x \neq 0$  and  $y \neq 0$ , then  $xy \neq 0$

inverses  
are  
unique

$$\textcircled{11} (-x)y = -(xy) = x(-y)$$

$$\textcircled{12} (-x)(-y) = xy$$

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proof:

① Suppose  $x+y = x+z$ .

Then,

$$y = 0 + y = (-x+x) + y = -x + (x+y)$$

$\uparrow$  (A4)       $\uparrow$  (A5)       $\uparrow$  (A3)

$$= -x + (x+z) = (-x+x) + z$$

$$= 0 + z = z$$

$\uparrow$  (A5)       $\uparrow$  (A4)

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② Plug  $z=0$  into part 1.

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③ Plug  $z=-x$  into part 1.

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④ Since  $(-x) + x = 0$

We can apply part 3 to get  $x = -(-x)$ .

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⑤ - ⑧: You try, they are similar to ①-④.

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⑨ We have

$$0x = (0+0)x = 0x + 0x$$

$\uparrow$  (A4)       $\uparrow$  (D1)

Then,

$$-0x + 0x = -0x + (0x + 0x)$$

So by (A5) and (A3) we get

$$0 = (-0x + 0x) + 0x$$

By (A5) we get

$$0 = 0 + 0x$$

By (A4) we get

$$0 = 0x.$$

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(10) Let  $x, y \in R$  with  $x \neq 0$  and  $y \neq 0$ .  
Let's show this implies that  $xy \neq 0$ .  
Suppose  $xy = 0$ .

Then

$$1 = 1 \cdot 1 = (\overset{\uparrow (M4)}{x^{-1}x})(\overset{\uparrow (M5)}{y^{-1}y}) = \overset{\uparrow (M3)}{x^{-1}(xy^{-1})}y = \overset{\uparrow (M2)}{x^{-1}(y^{-1}x)}y$$

$$= (\overset{\uparrow (M3)}{x^{-1}y^{-1}})(xy) = \overset{\uparrow (part\ 9)}{x^{-1}y^{-1}0} = 0$$

We get the contradiction  $1 = 0$ .

Hence we must have  $xy \neq 0$ .

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⑪ We have that

$$(-x)y + xy = (-x + x)y = 0y = 0$$

$\uparrow$  (D1)                       $\uparrow$  (A5)                       $\uparrow$  (part ⑨)

Thus, by part 3 we get

$$(-x)y = -(xy).$$

Similarly,

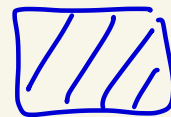
$$x(-y) + xy = x(-y + y) = x0 = 0$$

$$\text{So, } x(-y) = -(xy).$$

⑫ We have that

$$(-x)(-y) = -(x(-y)) = -(-(xy)) = xy$$

$\uparrow$  (part ⑪)                       $\uparrow$  (part ④)



Theorem: Let  $x, y \in \mathbb{R}$ .

Then:

① If  $x > 0$ , then  $-x < 0$ .

② If  $x < 0$ , then  $-x > 0$ .

③ If  $x > 0$  and  $y < z$ , then  $xy < xz$ .

④ If  $x < 0$  and  $y < z$ , then  $xy > xz$ .

⑤ If  $x \neq 0$ , then  $x^2 > 0$ .

⑥  $1 > 0$

⑦ If  $0 < x < y$ , then  $0 < y^{-1} < x^{-1}$ .

Proof:

① If  $x > 0$ , then

$$0 = \underset{\substack{\uparrow \\ (A5)}}{-x} + x \underset{\substack{\uparrow \\ (O3)}}{>} -x + 0 = -x$$

Thus,  $0 > -x$ .

② If  $x < 0$ , then

$$0 = \underset{\substack{\uparrow \\ (A5)}}{-x} + x \underset{\substack{\uparrow \\ (O3)}}{<} -x + 0 = -x.$$

Thus,  $0 < -x$ .

③ Suppose  $x > 0$  and  $y < z$ .

Then,

$$z - y > \underset{\substack{\uparrow \\ (03)}}{y - y} = 0$$

So,

$$z - y > 0.$$

Thus, by (04) we get

$$x(z - y) > 0$$

So,

$$\begin{aligned} xz &\stackrel{(A4)}{=} xz + 0 \\ &= xz + (-xy + xy) \\ &\stackrel{(A5)}{=} \underset{\substack{\uparrow \\ (A3)}}{xz - xy} + xy \\ &= \underset{\substack{\uparrow \\ (D1)}}{x(z - y)} + xy \\ &> \underset{\substack{\uparrow \\ (03)}}{0} + xy \\ &= \underset{\substack{\uparrow \\ (A4)}}{xy} \end{aligned}$$

Thus,  $xz > xy$ .

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④ Suppose  $x < 0$  and  $y < z$ .

Since  $x < 0$  we have  $-x > 0$ .  $\leftarrow$  (part 2)

Since  $y < z$  we have  $-y + y < -y + z \leftarrow (03)$

So,  $0 < z - y \leftarrow (A4) \text{ and } (A2)$

Then,

$$- [x(z-y)] = (-x)(z-y) > 0$$

$\uparrow$  (previous theorem)  $\uparrow$  (04)

Thus,  $x(z-y) < 0$  by part 1.

So,  $xz - xy < 0$ .  $\leftarrow$  (D1 and previous theorem)

Thus,  $xz - xy + xy < 0 + xy \leftarrow (03)$

So,  $xz + 0 < xy \leftarrow (A4, A5)$

Thus,  $xz < xy$ .  $\leftarrow$  (A4)

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⑤ Suppose  $x \neq 0$ .

If  $x > 0$ , then  $x \cdot x > 0 \cdot 0$  by (04).

So if  $x > 0$ , then  $x^2 > 0$ .

$0 \cdot 0 = 0$   
by prev.  
theorem



If  $x < 0$ , then  $-x > 0$  by part 2.  
Hence if  $-x < 0$ , then  $(-x)^2 > 0$ .  
But  $(-x)^2 = (-x)(-x) = x^2$  by prev. thm.  
Thus, if  $x < 0$ , then  $x^2 > 0$ .

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⑥ Plug  $x = 1$  into part 5  
to get that  $1^2 > 0$ . (M4)  
Thus,  $1 > 0$

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⑦ Suppose  $0 < x < y$ .  
By (02) we know  $0 < y$ .

We know  $y \cdot y^{-1} = 1 > 0$   
↑ (M5)      ↑ (part ⑥)

Since  $y > 0$  and  $yy^{-1} > 0$  we must  
have that  $y^{-1} > 0$  by (04).

Similarly  $x^{-1} > 0$ .

Since  $x < y$  we know  $x(\bar{x}^{-1}\bar{y}^{-1}) < y(\bar{x}^{-1}\bar{y}^{-1})$

Thus,  $(xx^{-1})\bar{y}^{-1} < y(\bar{y}^{-1}x^{-1})$  by (M3).

So,  $1\bar{y}^{-1} < (y\bar{y}^{-1})x^{-1}$  by (M3, M5)

Thus,  $\bar{y}^{-1} < 1x^{-1}$  by (M4, M5)

So,  $y^{-1} < x^{-1}$  by (M4).

Thus,  $0 < y^{-1} < x^{-1}$ .

